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A Theorem on the Uniform Convergence of Fourier-Type Integrals*

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A Dirichlet-type** theorem for the uniform convergence of integrals of the form

$$\int_0^\infty \cos(tf(\alpha)) \cos(x\alpha)\, d\alpha$$

is given. The theorem is illustrated by an application in the solution of a simple fourth order partial differential equation.

In general treatments of Fourier transforms it is usually assumed that the integrand $F: (-\infty, \infty) \to R^1$ satisfies one of the following conditions:

(a) F satisfies Dirichlet's conditions on R^1

(b) $F \in \mathscr{L}^p$, $p \in [1, 2]$, on R^1.

Theorems then involve these hypotheses. In individual cases, when these conditions are not satisfied, some knowledge concerning the uniform convergence of the integrals in question is a boon in the calculations. The following theorem then is to provide such an aid.

THEOREM. *Let $f: [0, \infty) \to R^1$ be in C^1 on $[0, \infty)$. Assume that the derivative, f', is unbounded and monotonically increasing for $\alpha \geqslant A_1 > 0$. Then*

$$I(x, t) = \lim_{R\to\infty} \int_0^R \cos(tf(\alpha)) \cos(x\alpha)\, d\alpha$$

converges uniformly for every

$$(x, t) \in S \triangleq [-M, M] \times [\gamma, \infty),$$

where $\gamma > 0$ and $M < \infty$.

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** See reference [1].

Proof. (1) *Preliminary Observations.* We first note that for every $R \in (0, \infty)$

$$\int_0^R \cos(tf(\alpha)) \cos(x\alpha)\, d\alpha$$
$$= \tfrac{1}{2} \int_0^R \cos(tf(\alpha) + x\alpha)\, d\alpha + \tfrac{1}{2} \int_0^R \cos(tf(\alpha) - x\alpha)\, d\alpha. \tag{1}$$

Thus, it suffices to consider the convergence of

$$\lim_{R\to\infty} \int_0^R \cos(tf(\alpha) + \alpha x)\, d\alpha = \lim_{R\to\infty} \int_0^R g(\alpha, x, t)\, d(h(\alpha, x, t)), \tag{2}$$

where we have defined

$$g(\alpha, x, t) = \frac{1}{tf'(\alpha) + x}, \qquad h(\alpha, x, t) = \sin(tf(\alpha) + \alpha x). \tag{3}$$

Let $\gamma > 0$ be given and let $x \in [-M, M]$. Choose $A_2 > 0$, such that $\gamma f'(\alpha) > |M|$ for $\alpha \geqslant A_2$. Furthermore, choose $\alpha_0 = \max[A_1, A_2]$. Then,

$$\begin{aligned} |g(\alpha_1, x, t)| &\leqslant |g(\alpha, x, t)| && \text{for } \alpha_1 \geqslant \alpha \geqslant \alpha_0 \\ |g(\alpha, x, t)| &\leqslant |g(\alpha, -M, \gamma)| && \forall (x, t) \in S \\ |h(\alpha, x, t)| &\leqslant 1 && \forall (\alpha, x, t) \in [0, \infty) \times S. \end{aligned} \tag{4}$$

(2) *Proof Proper.* We shall show that the integral satisfies the uniform Cauchy condition.

Let $\epsilon > 0$ be given. Let $A_3 > 0$ be such that

$$|g(u, -M, \gamma)| \leqslant \frac{\epsilon}{4} \qquad \text{for } u \geqslant A_3, \tag{5}$$

and choose $A_\epsilon = \max[\alpha_0, A_3]$ and $v > u \geqslant A_\epsilon$. Then we may write

$$\int_u^v g\,dh = g(v, x, t)\, h(v, x, t) - g(u, x, t)\, h(u, x, t) + \int_u^v h\,d(-g). \tag{6}$$

Since $(-g)$ is an increasing function of α for every $(x, t) \in S$, and since h is bounded on $[0, \infty) \times S$

$$\left| \int_u^v h\,d(-g) \right| \leqslant \int_u^v d(-g) = g(u, x, t) - g(v, x, t), \tag{7}$$

and it follows that

$$\begin{aligned}\left|\int_u^v g\,dh\right| &\leqslant g(v, x, t) + g(u, x, t) + g(u, x, t) + g(v, x, t)\\ &\leqslant 2g(u, -M, \gamma) + 2g(v, -M, \gamma)\\ &\leqslant 4g(u, -M, \gamma)\\ &\leqslant \epsilon.\end{aligned}$$

A similar argument applies with the minus sign in the integrand of (2). Thus, I satisfies a uniform Cauchy condition for $(x, t) \in S$, and consequently, converges uniformly for $(x, t) \in S$. Q.E.D.

Remarks. (1) Convergence is not uniform for $t > 0$. For, with

$$f(\alpha) = \alpha^2, \qquad x = 0, \qquad t = 1/n, \qquad v = (n\pi)^{1/2}, \qquad u = (n\pi/2)^{1/2},$$

we have

$$\left|\int_u^v \cos(t\alpha^2)\,d\alpha\right| = (n\pi/2)^{1/2}\,|\mathscr{C}(\sqrt{2}) - \mathscr{C}(1)| = (n\pi/2)^{1/2}\,k > k \quad \text{for } n \geqslant 1,$$

so that the uniform Cauchy condition cannot be satisfied. $\mathscr{C}$ here is the Fresnel integral.

(2) I is continuous on $(-\infty, \infty) \times (0, \infty)$. Let $t_0 > 0$. Choose γ such that $t_0 > \gamma > 0$. Since I converges uniformly for $t \geqslant \gamma$, I is continuous at t_0. Since t_0 was arbitrary, it follows that I is continuous for $t > 0$. An analogous argument may be used for x.

(3) The theorem applies if f is any polynomial

$$f(\alpha) = a_n\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0\,,$$

with $a_n > 0$, $n > 1$.

(4) Naturally the theorem applies with combinations of sines and cosines is the integrand.

Example. For simplicity we illustrate the theorem for a problem with known solution.

Solve

$$\frac{\partial^4 w(x, t)}{\partial x^4} + \frac{\partial^2 w(x, t)}{\partial t^2} = 0 \qquad \text{on} \qquad (-\infty, \infty) \times (0, \infty),$$

subject to

(i) the initial conditions,

$$w(x, 0^+) = f(x), \qquad \frac{\partial w(x, 0^+)}{\partial t} = g(x),$$

(ii) the order requirements,

$$\lim_{|x|\to\infty} \frac{\partial^\nu w(x, t)}{\partial x^\nu} = 0, \qquad \nu = 0, 1, 2, 3;$$

(iii) in addition we assume that the usual hypotheses for the existence of Fourier- and Laplace-transforms are satisfied.

If we denote the Fourier- and Laplace-transforms respectively by

$$\mathscr{F}\{w(x, t)\} = W(\alpha, t) \qquad \text{and} \qquad \mathscr{L}\{w(x, t)\} = \bar{w}(x, s),$$

we may solve for the double transform of $w(x, t)$ as

$$\overline{W}(\alpha, s) = \frac{G(\alpha) + sF(\alpha)}{s^2 + \alpha^4},$$

with inverse $\mathscr{L}$-transform

$$W(\alpha, t) = G(\alpha) \frac{\sin(t\alpha^2)}{\alpha^2} + F(\alpha) \cos(t\alpha^2).$$

We have

$$h(x, t) \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin(t\alpha^2)}{\alpha^2} e^{i\alpha x}\, d\alpha = \frac{1}{(4\pi)^{1/2}} \int_0^t \frac{1}{(u)^{1/2}} \cos\left(\frac{x^2}{4u} - \frac{\pi}{4}\right) du,$$

and in view of the previous theorem

$$h_t(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos(t\alpha^2)\, e^{i\alpha x}\, d\alpha = \frac{1}{(4\pi t)^{1/2}} \cos\left(\frac{x^2}{4t} - \frac{\pi}{4}\right),$$

so that the solution of the problem is

$$w(x, t) = \int_{-\infty}^{\infty} g(x - \eta)\, h(\eta, t)\, d\eta + \int_{-\infty}^{\infty} f(x - \eta)\, h_t(\eta, t)\, d\eta.$$

Reference

1. T. M. Apostol, "Mathematical Analysis," Addison-Wesley, Reading, MA, 1960